SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 7]*

Let $A = \begin{pmatrix} 5 & 1 \\ 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix}$.

(a) (i) Find AB. $= \boxed{\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}}$

(ii) Write down the inverse of A. *[3 marks]*

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{2}{4} & -\frac{1}{4} \\ -\frac{6}{4} & \frac{5}{4} \end{pmatrix}}$$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$.

(b) Solve the matrix equation $AX = C$. *[4 marks]*

$$\begin{pmatrix} \frac{2}{4} & -\frac{1}{4} \\ -\frac{6}{4} & \frac{5}{4} \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{4} & -\frac{1}{4} \\ -\frac{6}{4} & \frac{5}{4} \end{pmatrix}\begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ -17 \end{pmatrix}}$$

March 21, 2014

MATHEMATICS
STANDARD LEVEL
PAPER 1

Thursday 7 May 2009 (afternoon)

1 hour 30 minutes

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2. *[Maximum mark: 6]*

The letters of the word PROBABILITY are written on 11 cards as shown below.

P	R	O	~~B~~	~~A~~	~~B~~	I	L	I	T	Y

Two cards are drawn at random without replacement.
Let A be the event the first card drawn is the letter A.
Let B be the event the second card drawn is the letter B.

(a) Find $\mathrm{P}(A)$. $\boxed{\frac{1}{11}}$ *[1 mark]*

(b) Find $\mathrm{P}(B \mid A)$. $\frac{2}{10} = \boxed{\frac{1}{5}}$ *[2 marks]*

Also,
$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{11} \cdot \frac{1}{5}}{\frac{1}{11}}$$

(c) Find $\mathrm{P}(A \cap B)$. *[3 marks]*

$$\frac{1}{11} \cdot \frac{1}{5} = \boxed{\frac{1}{55}}$$

$P(A \cap B)$ = Both events occured
$P(A \cup B)$ = either (or both) occured

3. *[Maximum mark: 6]*

Let $f(x) = e^x \cos x$. Find the gradient of the normal to the curve of f at $x = \pi$.

$f'(x) = e^x(-\sin x) + \cos x \, e^x$ (product rule)

$f'(\pi) = e^\pi(-\sin \pi) + \cos \pi \, e^\pi$

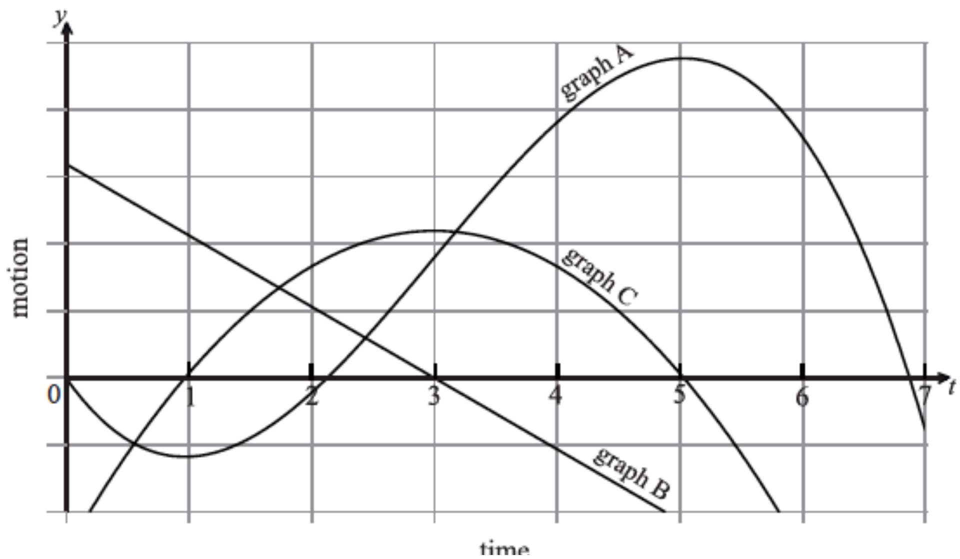
$= -e^\pi$

opposite reciprocal: $\dfrac{1}{e^\pi}$

Why?

4. *[Maximum mark: 6]*

The following diagram shows the graphs of the **displacement**, **velocity** and **acceleration** of a moving object as functions of time, t.



(a) Complete the following table by noting which graph A, B or C corresponds to each function. *[4 marks]*

Function	Graph
displacement	A
acceleration	B

(b) Write down the value of t when the velocity is greatest. $t=3$ *[2 marks]*

5. *[Maximum mark: 6]*

Let $f(x) = x^2$ and $g(x) = 2(x-1)^2$.

(a) The graph of g can be obtained from the graph of f using two transformations. Give a full geometric description of each of the two transformations. *[2 marks]*

move f 1 right

and stretch vertically by factor 2

(b) The graph of g is translated by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to give the graph of h.

The point $(-1, 1)$ on the graph of f is translated to the point P on the graph of h. Find the coordinates of P. *[4 marks]*

$h(x) = 2(x-4)^2 - 2$

-1 is moved 1 right $\rightarrow 0$

0 is moved 3 right $\rightarrow 3$

* $h(3) = 2(3-4)^2 - 2$

$P = (3, 0)$

* also,

1 stretched vertically factor 2 $\rightarrow 2$

2 moved 2 down is 0

6. *[Maximum mark: 7]*

Let $f(x) = e^{x+3}$.

(a) (i) Show that $f^{-1}(x) = \ln x - 3$.

$x = e^{y+3}$

$\ln x = y + 3$

$f^{-1}(x) = \ln x - 3$

(ii) Write down the domain of f^{-1}. $(0, \infty)$ *[3 marks]*

(b) Solve the equation $f^{-1}(x) = \ln\left(\frac{1}{x}\right)$. *[4 marks]*

$\ln x - 3 = \ln \frac{1}{x}$

$\ln x - 3 = \ln x^{-1}$

$\ln x - \ln x^{-1} = 3$

$\ln \frac{x}{x^{-1}} = 3$

$\ln x^2 = 3$

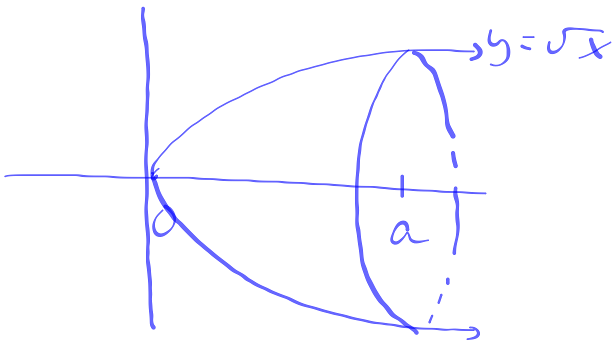
$e^3 = x^2$

$\pm\sqrt{e^3} = x$

$\sqrt{e^3} = x$

7. *[Maximum mark: 7]*

The graph of $y=\sqrt{x}$ between $x=0$ and $x=a$ is rotated 360° about the x-axis. The volume of the solid formed is 32π. Find the value of a.



$$V = \int \pi y^2 \, dx$$

$$\pi \int_0^a (\sqrt{x})^2 \, dx$$

$$\pi \int_0^a x$$

$$\pi \left[\left(\frac{x^2}{2} \right) \right]_0^a$$

$$\frac{a^2}{2} - \frac{0^2}{2}$$

$$\frac{\pi a^2}{2} = 32\pi$$

$$a^2 = 64$$

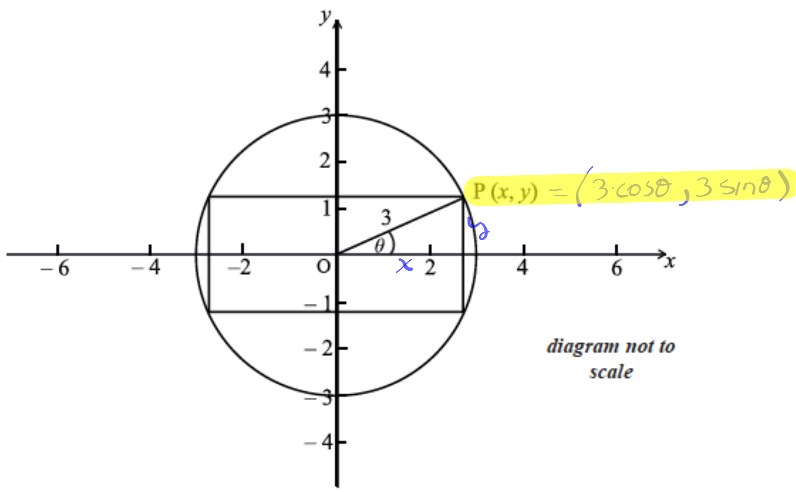
$$\boxed{a = 8}$$

SECTION B

Answer all the questions on the answer sheets provided. Please start each question on a new page.

8. *[Maximum mark: 13]*

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point $P(x, y)$ is a vertex of the rectangle and also lies on the circle. The angle between (OP) and the x-axis is θ radians, where $0 \le \theta \le \frac{\pi}{2}$.

(a) Write down an expression in terms of θ for

(i) x; $\cos\theta = \frac{x}{3}$ $x = 3\cos\theta$

(ii) y. $y = 3\sin\theta$ *[2 marks]*

Let the area of the rectangle be A.

(b) Show that $A = 18\sin 2\theta$. *[3 marks]*

$A_{\square} = (2 \cdot 3\cos\theta)(2 \cdot 3\sin\theta)$

$= 36\cos\theta\sin\theta$

$= 18 \cdot (2\cos\theta\sin\theta)$

$18(\sin 2\theta)$

(c) (i) Find $\frac{dA}{d\theta}$.

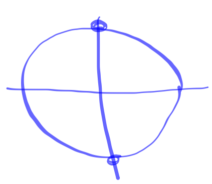
(ii) Hence, find the exact value of θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of θ does give a maximum. *[8 marks]*

c) i) $\frac{dA}{d\theta} = 36\cos 2\theta$

ii) $36\cos 2\theta = 0$

$\cos 2\theta = 0$



$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}$

iii)

@ $\frac{\pi}{4}$,

$\frac{d^2A}{d\theta^2} < 0$

max when

$f''(x) < 0$

9. *[Maximum mark: 16]*

The vertices of the triangle PQR are defined by the position vectors

$$\vec{OP} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \ \vec{OQ} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \text{ and } \vec{OR} = \begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix}.$$

(a) Find

(i) $\vec{PQ}$;

(ii) $\vec{PR}$.

i) $\vec{PQ} = \vec{PO} + \vec{OQ}$

$\begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

ii) $\vec{PR} = \vec{PO} + \vec{OR}$

$= \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$

(b) Show that $\cos R\hat{P}Q = \frac{1}{2}$. *[7 marks]*

To find $\angle$ between 2 vectors.

$$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|v||w|}$$

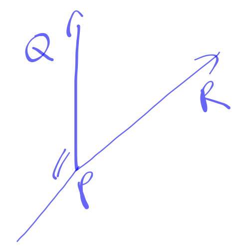
$$\frac{(-1)(2)+(2)(2)+(1)(4)}{\sqrt{(-1)^2+2^2+1^2}\sqrt{2^2+2^2+4^2}}$$

$$\frac{6}{\sqrt{6}\sqrt{24}}$$

$$\frac{6}{\sqrt{144}}$$

$$\boxed{\frac{1}{2}}$$

$\vec{PQ}$ 3 $\vec{PR}$ must start at P !!



(c) (i) Find $\sin R\hat{P}Q$.

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{1}{2}\right)^2 = 1$

$\sin^2\theta = \frac{3}{4}$

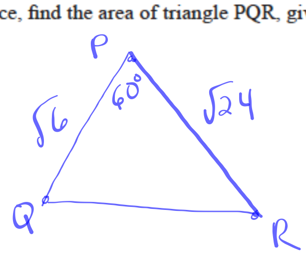
$\sin\theta = \frac{\pm\sqrt{3}}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$

since $\cos\theta = \frac{1}{2}$

$\theta = 60°$ or $\cancel{300}$

(ii) Hence, find the area of triangle PQR, giving your answer in the form $a\sqrt{3}$. *[6 marks]*



Area $= \frac{1}{2} \cdot \sqrt{6}\sqrt{24} \sin 60$

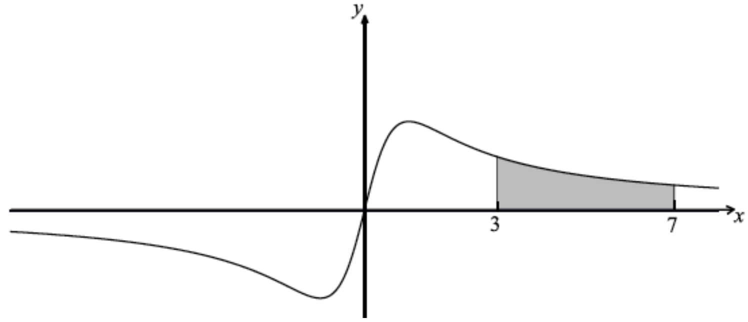
$\frac{1}{2} \cdot 12 \cdot \frac{\sqrt{3}}{2}$

$\frac{6\sqrt{3}}{2}$

$\boxed{3\sqrt{3}}$

10. *[Maximum mark: 16]*

Let $f(x)=\frac{ax}{x^2+1}$, $-8\le x\le 8$, $a\in\mathbb{R}$. The graph of f is shown below.



The region between $x=3$ and $x=7$ is shaded.

(a) Show that $f(-x)=-f(x)$. *[2 marks]*

$f(-x)=\frac{-ax}{(-x)^2+1}$

$=-\frac{ax}{x^2+1}$

$=-f(x)$

(b) Given that $f''(x)=\frac{2ax(x^2-3)}{(x^2+1)^3}$, find the coordinates of all points of inflexion. *[7 marks]*

$\frac{2ax(x^2-3)}{(x^2+1)^3}=0$

$2ax(x^2-3)=0$

$2ax=0$, $x^2-3=0$

$x=0$ $\quad x^2=3$

$x=\pm\sqrt{3}$

$\boxed{(0,0)\left(\sqrt{3},\frac{a\sqrt{3}}{4}\right)\left(-\sqrt{3},\frac{-a\sqrt{3}}{4}\right)}$

(c) It is given that $\int f(x)\,dx=\frac{a}{2}\ln(x^2+1)+C$.

(i) Find the area of the shaded region, giving your answer in the form $p\ln q$.

$\left[\frac{a}{2}\ln(x^2+1)\right]_3^7$

$\frac{a}{2}\ln 50-\frac{a}{2}\ln 10$

$\frac{a}{2}(\ln 50-\ln 10)$

$\boxed{\frac{a}{2}\ln 5}$

(ii) Find the value of $\int_4^8 2f(x-1)\,dx$. *[7 marks]*

$(x-1)$ shift doesn't change area, but the 2 doubles the area!

$2\int_3^7 f(x)$

$2\cdot\frac{a}{2}\ln 5$

$\boxed{a\ln 5}$